Some harder examples of subsidiary angle form in trigonometry

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1. Find the maximum and minimum of $\dfrac{1+\sin\theta}{3-\cos\theta}$

Solution Let $y = \dfrac{1+\sin\theta}{3-\cos\theta}$.

$3y - y\cos\theta = 1 + \sin\theta$

$\sin\theta + y\cos\theta = 3y - 1$

$$\frac{1}{\sqrt{1+y^2}}\sin\theta + \frac{y}{\sqrt{1+y^2}}\cos\theta = \frac{3y-1}{\sqrt{1+y^2}} \qquad \ldots. \quad (1)$$

Let $\cos\alpha = \dfrac{1}{\sqrt{1+y^2}}$, $\sin\alpha = \dfrac{y}{\sqrt{1+y^2}}$

(1) becomes, $\sin\theta\cos\alpha + \cos\theta\sin\alpha = \dfrac{3y-1}{\sqrt{1+y^2}}$

$\sin(\theta-\alpha) = \dfrac{3y-1}{\sqrt{1+y^2}}$ (☺ **subsidiary angle form in L.H.S.)**

Since $|\sin(\theta-\alpha)| \le 1$, therefore $\left|\dfrac{3y-1}{\sqrt{1+y^2}}\right| \le 1$

$(3y-1)^2 \le 1 + y^2$

$9y^2 - 6y + 1 \le 1 + y^2$

$8y^2 - 6y \le 0$

$y(4y-3) \le 0$

$\therefore$ $0 \le y \le \dfrac{3}{4}$ or $0 \le \dfrac{1+\sin\theta}{3-\cos\theta} \le \dfrac{3}{4}$.

$\therefore$ Min. of $\dfrac{1+\sin\theta}{3-\cos\theta} = 0$, Max. of $\dfrac{1+\sin\theta}{3-\cos\theta} = \dfrac{3}{4}$.

Note 1. If you are given to prove : $0 \le \dfrac{1+\sin\theta}{3-\cos\theta} \le \dfrac{3}{4}$

Solution $1 + \sin\theta \ge 0$ and $3 - \cos\theta \ge 3 - 1 > 0$ $\Rightarrow$ $0 \le \dfrac{1+\sin\theta}{3-\cos\theta}$

Also, $4\sin\theta + 3\cos\theta = 5\sin(\theta+\alpha) \le 5$, where $\alpha = \tan^{-1}\left(\dfrac{3}{4}\right)$.

$\therefore$ $4 + 4\sin\theta \le 9 - 3\cos\theta$ $\Rightarrow 4(1+\sin\theta) \le 3(3-\cos\theta)$

As $3 - \cos\theta > 0$, we have $\dfrac{1+\sin\theta}{3-\cos\theta} \le \dfrac{3}{4}$. $\therefore$ $0 \le \dfrac{1+\sin\theta}{3-\cos\theta} \le \dfrac{3}{4}$

2. This is also a good question of application of differentiation in Calculus. ☺

2. Solve : $8\cos x = \frac{\sqrt{3}}{\sin x} + \frac{1}{\cos x}$

Solution

Multiply the given equation by $\sin x \cos x$, we have:

$$8\sin x\cos^2 x = \sqrt{3}\cos x + \sin x$$

$$2\sin 2x\cos x = \frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x$$

$$\sin 3x + \sin x = \frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x$$

$$\sin 3x = \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x$$

$$\sin 3x = \sin\frac{\pi}{3}\cos x - \cos\frac{\pi}{3}\sin x$$

$$\sin 3x = \sin\left(\frac{\pi}{3} - x\right)$$ (☺ **subsidiary angle form in R.H.S.)**

$\therefore$ $3x = 2n\pi + \left(\frac{\pi}{3} - x\right)$ or $3x = 2n\pi + \pi - \left(\frac{\pi}{3} - x\right)$, where n is an integer.

$\therefore$ $x = \frac{1}{2}n\pi + \frac{\pi}{12}$ or $x = n\pi + \frac{\pi}{3}$, where n is an integer.

For the roots above , we check that $\sin x \cos x \neq 0$ and they are roots of the given equation .

3. Solve : $\sin(\pi\cos x) = \cos(\pi\sin x)$

Solution

$$\sin(\pi\cos x) = \cos(\pi\sin x)$$

$$\sin(\pi\cos x) = \sin\left[\frac{\pi}{2} - (\pi\sin x)\right]$$

$\therefore$ $\pi\cos x = 2n\pi + \left[\frac{\pi}{2} - (\pi\sin x)\right]$ or $\pi\cos x = 2n\pi + \pi - \left[\frac{\pi}{2} - (\pi\sin x)\right]$, where n is an integer.

$\cos x \pm \sin x = 2n + \frac{1}{2}$, where n is an integer.

Now, since $|\cos x \pm \sin x| = \sqrt{2}\left|\cos\left(x \pm \frac{\pi}{4}\right)\right| \le \sqrt{2}$ (☺ **subsidiary angle form)**

$\therefore$ The only value for n is 0 .

The equation becomes : $\sqrt{2}\cos\left(x \pm \frac{\pi}{4}\right) = \frac{1}{2}$ or $\cos\left(x \pm \frac{\pi}{4}\right) = \frac{\sqrt{2}}{4}$

$x \pm \frac{\pi}{4} = 2k\pi \pm \cos^{-1}\frac{\sqrt{2}}{4}$ (use k here because n is used once)

$\therefore$ $x = \left(2k \pm \frac{1}{4}\right)\pi \pm \cos^{-1}\frac{\sqrt{2}}{4}$,where k is an integer.

4. **(a)** Given that k is a constant, show that $y = \sin\left(\frac{\pi}{2} - x\right) + k\cos\left(\frac{\pi}{3} - x\right)$

can be expressed in the form $y = r\sin(x + \alpha)$, where $r > 0$.

(b) What is the value of k when $\alpha = \frac{2}{3}\pi$? Plot its graph for this value of α.

Solution

(a) $y = \sin\left(\frac{\pi}{2} - x\right) + k\cos\left(\frac{\pi}{3} - x\right) = \cos x + k\left(\cos\frac{\pi}{3}\cos x + \sin\frac{\pi}{3}\sin x\right)$

$$= \frac{\sqrt{3}}{2}k\sin x + \left(1 + \frac{k}{2}\right)\cos x$$

$$= \sqrt{\frac{3}{4}k^2 + 1 + k + \frac{k^2}{4}}\left(\frac{\sqrt{3}k}{2\sqrt{k^2+k+1}}\sin x + \frac{k+2}{2\sqrt{k^2+k+1}}\cos x\right)$$

$$= \sqrt{k^2+k+1}\sin(x+\alpha)$$ (☺ **subsidiary form)**

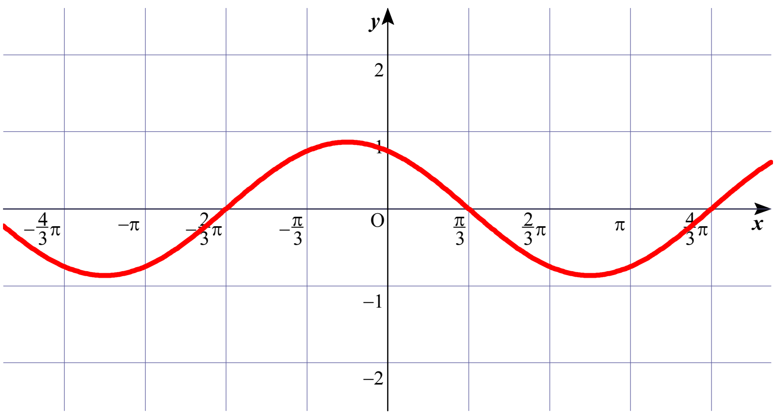
where $r = \sqrt{k^2+k+1}$ and $\tan\alpha = \frac{k+2}{\sqrt{3}k}$.

(b) When $\alpha = \frac{2}{3}\pi$, $\tan\frac{2}{3}\pi = -\sqrt{3} = \frac{k+2}{\sqrt{3}k}$.

$-3k = k + 2$, $\therefore$ $k = -\frac{1}{2}$

Then $y = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) + 1}\,\sin\left(x + \frac{2}{3}\pi\right) = \frac{\sqrt{3}}{2}\sin\left(x + \frac{2}{3}\pi\right)$

Graph :



Exercise

1. Find the relative maximum and relative minimum of $\frac{2\cos\theta + 1}{2\cos\theta - 1}$.

(Answer : relative maximum = $\frac{1}{3}$, relative minimum = 3)

2. Sketch the graph : $y = \sin x + \sqrt{3}\cos x$.